[3] See *ibid.*, **40**, 1078, 1954, hereafter cited as "RVI."

[4] See RVI. The last statement of RVI is not correct. In order to rectify it, we have to replace π_ξ there by the definition of π_Λ given above in the present note.

[5] H. Weyl, *Math. Z.*, **24**, 328–395, 1925.

[6] J. von Neumann, *Ann. Math.*, **50**, 401–485, 1949.

[7] F. I. Mautner, *Ann. Math.*, **52**, 528–555, 1950.

[8] *Trans. Am. Math. Soc.*, **75**, 230, 1953, Theorem 7.

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*A FUNDAMENTAL INEQUALITY IN THE THEORY OF PICARD VARIETIES**

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1. *Introduction.*—In abstract algebraic geometry we can define two kinds of irregularities of an algebraic variety, and also one more if the variety is normal. They are, substantially, of different natures, and one of the main theorems in transcendental algebraic geometry asserts that they have the same value if the variety is nonsingular. In this paper we shall prove an inequality among two of them, i.e., we shall show that the dimension q of the Picard variety of a variety V is not greater than the dimension $h^{1,0}$ of the space of linear differential forms of the first kind on V. Actually, we shall prove more, i.e., the following: *Let (A, f) be the Albanese variety of V. Then δf has an "isomorphism property" in the sense that δf gives an isomorphism of the space of linear differential forms of the first kind on A into the corresponding space associated with V.* Since the linear differential forms of the first kind on an Abelian variety are exactly the Maurer-Cartan forms, it follows that the dimension of A, which is equal to q by duality,[1] is not greater than $h^{1,0}$.

2. *Summary of General Results.*—Let f be a function on a variety V with values in a variety U. Let K be a field of definition of f, hence also of V, and let M be a generic point of V over K. If $K(M)$ is separably generated over $K(f(M))$, then f is called *separable.* It can be verified easily that this property is "absolute," i.e., it is independent of the choice of K and M. If the image variety W is not contained in the singular locus of U, we can define a mapping δf of the space of differential forms on U which are "regular along W" into the space of all differential forms on V.[2] The mapping δf is linear over the universal domain $\boldsymbol{K}$ and is commutative with the exterior multiplication and the exterior differentiation of differential forms. Moreover, if ϕ is a numerical function on U which is regular along W, then $\delta f \cdot \phi = \phi \circ f$ holds. Here $\phi \circ f$ denotes the composite function of f and ϕ, which is defined without ambiguity by assumption. Conversely, these properties determine δf uniquely. It is easy to see that f, regarded as a function on V with values in W, is separable if and only if δf gives an isomorphism of the space of all differential forms on W into the corresponding space associated with V. On the other hand, let g be a function on U with values in a variety T. If g is regular along W, and if θ is a differential form on T which is regular along the image variety of g, then we have $\delta f[\delta g \cdot \theta] = \delta(g \circ f) \cdot \theta$. In particular, if f is an inclusion map of V into

U, we have $\mathrm{Tr}_V\ [\delta g \cdot \theta] = \delta(\mathrm{Tr}_V\ g) \cdot \theta$. A differential form on U is said to be *of the first kind* if it is regular at every simple point of any model of the function field of U.[3] If U is nonsingular, the differential form is already of the first kind if it is everywhere regular on U. More generally, if U is nonsingular, δf maps the space of differential forms on U which are everywhere regular on U into the space of differential forms of the first kind on V.[4] In particular, this is always the case if U is an Abelian variety. On an Abelian variety A, a differential form is everywhere regular if and only if it is invariant by δt for all translations t of A. There are as many linearly independent linear differential forms of the first kind on A as its dimension. We know also that every invariant differential form on A is closed, at least if the characteristic of $\boldsymbol{K}$ is different from 2.[5]

3. *The Isomorphism Property of δf.*—Let V be a nonsingular projective model of a function field of dimension 2 over $\boldsymbol{K}$. We have constructed elsewhere[6] a linear pencil $\{C_u\}$ on V with a parameter straight line D of the "simplest possible nature." We have also defined a subvariety $\boldsymbol{J}$ of a product $D \times L^N$ of D and of a projective space L^N such that $\boldsymbol{J} \cdot ((u) \times L^N) = (u) \times J_u$ is defined for every (u) and such that J_u is the Jacobian variety of C_u, when C_u is nonsingular, and J_u is a completion of the "generalized Jacobian variety" of C_u, when C_u has a double point. Moreover, we have defined a birational correspondence φ of V into $\boldsymbol{J}$ which has the following property:

LEMMA 1. *The mapping $\delta\varphi$ gives an isomorphism of the space of linear differential forms of the first kind on $\boldsymbol{J}$ onto the corresponding space associated with V.*

On the other hand, let (A, h) be the Albanese variety of $\boldsymbol{J}$. Since h is regular at every simple point of $\boldsymbol{J}$,[7] the composite function $f = h \circ \varphi$ is defined. Moreover, as we have also shown, (A, f) is the Albanese variety of V:

$$\begin{tikzcd} V \arrow[rr, "\varphi"] \arrow[dr, "f"'] & & \boldsymbol{J} \arrow[dl, "h"] \\ & A & \end{tikzcd}$$

Therefore, by Lemma 1, δf has an isomorphism property if and only if δh has an isomorphism property.

Now let m be a positive integer and consider the m-fold direct product $\boldsymbol{J} \times \ldots \times \boldsymbol{J}$. If $z_1, \ldots, z_m$ are m simple points of $\boldsymbol{J}$, we can define a function H on this product with values in A by $H(z_1 \times \ldots \times z_m) = \sum_{i=1}^{m} h(z_1)$. If we denote by $\boldsymbol{J}_u$ the product $(u) \times J_u$, then a result of Chow can be stated as follows:[8]

LEMMA 2. *Let $(u_1), \ldots, (u_m)$ be independent generic points of D over a field of definition K of $\boldsymbol{J}$, A and h. Then $Tr_{\boldsymbol{J}_{u_1} \times \ldots \times \boldsymbol{J}_{u_m}} H$ is separable for m sufficiently large.*

We shall now show that δh has an isomorphism property. Let θ be a linear differential form of the first kind on A. Then $\delta H \cdot \theta$ is a similar form on $\boldsymbol{J} \times \ldots \times \boldsymbol{J}$. Therefore, if p_i denotes the projection of $\boldsymbol{J} \times \ldots \times \boldsymbol{J}$ to its ith factor, there are uniquely determined m linear differential forms ω_i of the first kind on $\boldsymbol{J}$ such that $\delta H \cdot \theta = \sum_{i=1}^{m} \delta p_i \cdot \omega_i$ holds.[9] Since $\delta H \cdot \theta$ is invariant under the interchange of factors of the product, the ω_i coincide with each other. Hence we can write $\delta H\cdot\theta = \sum_{i=1}^{m}\delta p_i\cdot\omega$ with a single ω. Here $\omega = 0$ implies $\delta H\cdot\theta = 0$, hence, in particular, $\mathrm{Tr}_{J_{u_1}\times\ldots\times J_{u_m}}[\delta H\cdot\theta] = \delta(\mathrm{Tr}_{J_{u_1}\times\ldots\times J_{u_m}} H)\cdot\theta = 0$. However, since A itself is the image variety of $\mathrm{Tr}_{J_{u_1}\times\ldots\times J_{u_m}} H$, we conclude from Lemma 2 that $\theta = 0$. On the other hand, if we fix $m-1$ simple points $z_2,\ldots,z_m$ on J, and if z is a variable point on J, we can define a function g on J with values in $J\times\ldots\times J$ by $g(z) = z\times z_2\times\ldots\times z_m$. Also, let t be the translation of A associated with the point $\sum_{i=2}^{m} h(z_i)$ of A. Since θ is a Maurer-Cartan form, we then get $\delta g[\delta H\cdot\theta] = \delta(H\circ g)\cdot\theta = \delta(t\circ h)\cdot\theta = \delta h[\delta t\cdot\theta] = \delta h\cdot\theta$. On the other hand, we have $\delta g[\sum_{i=1}^{m}\delta p_i\cdot\omega] = \sum_{i=1}^{m}\delta g[\delta p_i\cdot\omega] = \sum_{i=1}^{m}\delta(p_i\circ g)\cdot\omega = \delta(p_1\circ g)\cdot\omega = \omega$. Hence we get $\omega = \delta h\cdot\theta$, and therefore δh has the isomorphism property.

On the other hand, if V is any algebraic surface, by a result of Zariski together with a recent result of Abhyankar,[10] we can find a nonsingular surface V' which is birationally equivalent to V. Let φ be the function on V with values in V' which represents this birational correspondence. Also, let (A, f') be the Albanese variety of V'. Then $f = f'\circ\varphi$ is defined, and (A, f) is the Albanese variety of V. Since $\delta f'$ has an isomorphism property, δf also has this property.

Finally, let V be an arbitrary variety, and let (A, f) be the Albanese variety of V. In order to prove the isomorphism property of δf, we may assume that $\dim(V) \geq 3$. Let K be a field of definition of A and f, hence also of V, and let L be a generic hyperplane over K in the ambient space of V. Then the intersection product $\bar{V} = V\cdot L$ is defined and is an absolutely irreducible variety of dimension $\dim(V) - 1$ by the "theorem of Bertini."[11] Moreover, if we put $\bar{f} = \mathrm{Tr}_{\bar{V}} f$, then we know that $(A, \bar{f})$ is the Albanese variety of $\bar{V}$.[12] Therefore, if we apply an induction to the dimension of V, we may assume that $\delta\bar{f} = \mathrm{Tr}_{\bar{V}}[\delta f]$ has an isomorphism property. Then, a fortiori, δf has the isomorphism property.

In this way the theorem we have stated in the Introduction is proved completely. We note also that the image by δf of the space of linear differential forms of the first kind on A is composed of closed forms, at least if the characteristic of K is different from 2.

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1 T. Matsusaka, "Some Theorems on Abelian Varieties," *Natl. Sci. Rept. Ochanomizu University*, **4**, 22–35, 1953.

2 Cf. S. Koizumi, "On the Differential Forms of the First Kind on Algebraic Varieties. I and II," *J. Math. Soc. Japan*, **1**, 273–280, 1949; **2**, 267–269, 1951.

3 This definition was given "tentatively" by A. Weil, in *Foundations of Algebraic Geometry* ("Am. Math. Soc. Colloquium Publs.," Vol. **29**, 1946), p. 246.

4 Koizumi, *op. cit.*, **1**, 277.

5 S. Nakano, "On Invariant Differential Forms on Group Varieties," *J. Math. Soc. Japan*, **2**, 216–227, 1951. Cf. also C. Chevalley, *Theory of Lie Groups*, Vol. I (Princeton, N.J.: Princeton University Press, 1946).

6 J. Igusa, "Fibre system of Jacobian varieties," *Am. J. Math.* (to appear).

7 A. Weil, *Variétés Abéliennes et courbes algébriques* ("Actualités Sci. et Ind.," No. 1064 [Paris: Hermann et Cie, 1948]), p. 27, Theorem 6.

[8] W. L. Chow, "Abelian Varieties over Function Fields," *Trans. Am. Math. Soc.* (to appear); the proof of the result quoted in the text is contained in a separate note, entitled "On Abelian Varieties over Function Fields," which will soon appear in print.

[9] Koizumi, *op. cit.*, **2**, 267.

[10] O. Zariski, "The Reduction of the Singularities of an Algebraic Surface," *Ann. Math.*, **40**, 639–689, 1939; "A Simplified Proof for the Resolution of Singularities of an Algebraic Surface," *ibid.*, **43**, 583–593, 1942; S. Abhyankar, "The Theorem of Local Uniformization on Algebraic Surfaces over Modular Fields," *Ann. Math.* (to appear).

[11] O. Zariski, "Pencils on an Algebraic Variety and a New Proof of a Theorem of Bertini," *Trans. Am. Math. Soc.*, **50**, 48–70, 1941; T. Matsusaka, "The Theorem of Bertini on Linear Systems in Modular Fields," *Kyoto Math. Mem.*, **26**, 51–62, 1950.

[12] Cf. Chow, "Abstract Theory of Picard Varieties," to appear soon in print.

A PATH SPACE AND THE STIEFEL-WHITNEY CLASSES

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The path space described here provides a very simple demonstration of the topological invariance of the Stiefel-Whitney classes for a differentiable manifold M. This invariance has been proved by other means by R. Thom.[1]

Actually, Thom proves a more general result, which implies the topological invariance of these classes. He shows that the fiber homotopy type of the tangent bundle of M depends only on the topological structure of M. Because the Stiefel-Whitney classes are dependent only on the fiber homotopy type of the tangent bundle, their topological invariance follows.

The path space we introduce is regarded as a fiber space[2] over M, and it turns out to have the same fiber homotopy type as the tangent bundle. Since the definition of the path space is purely topological, the general result of Thom follows immediately. Also, one sees directly that the Stiefel-Whitney classes have an analogue for manifolds without differentiability structure.

The paths considered are all continuous parametrized paths in M (parametrized by t, $0 \leq t \leq 1$), which do not recross the starting point (where $t = 0$). So, if $x(t)$ is the point with parameter t,

$$x(t) \neq x(0) \qquad \text{for} \qquad t > 0$$

is the requirement. These paths form a fiber space over M if we define the projection mapping to M by mapping each path into its starting point, $x(0)$.

We can regard M as provided with a smooth Riemannian metric. It is convenient[2] to assume that this metric is such that the geodesic distance between any pair of conjugate points is always more than one. Then, if two points are not more than one unit apart, there is a unique shortest geodesic segment joining them, and this segment varies continuously with the points.

The tangent bundle can now be regarded as formed by the geodesic paths of length 1, parametrized by arc length. This makes it a subspace of our fiber space